Composition and products in vector-valued Sobolev spaces and application

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Abstract

In this paper we will show the properties of composition operators $u \rightarrow f(u)$ in framework of E-valued Sobolev and Lizorkin-Triebel spaces. Here, E is a Banach space. Boundedness and continuity properties will be discussed in a certain detail in Sobolev-Lions type function space concerning two abstract spaces E_0 and E in terms of their interpolation. By using these composition properties, we obtain the local and global existence, uniqueness, and L^p-regularity of some nonlinear abstract diffusion equations.

1. Introduction and backgrounds

The boundedness and continuity properties of product and composition functions in different functional spaces were studied e.g. in [1-4, 8-9] and the references therein. Here, we show the composition and products properties in abstract Sobolev, Lizorkin-Triebel, and Sobolev-Lions spaces. The main motivation in proving our results comes from the study for analysing composition and products in fractional Sobolev spaces comes from the study of nonlinear evolution equations, see e.g.[7], [10] and the references therein.

In order to state our results precisely, we introduce some notations and some function spaces. Let E be a Banach space. $L^p(\Omega; E)$ denotes the space of strongly measurable E-valued functions that are defined on the measurable subset $\Omega \subset \mathbb{R}^n$ with the norm

$$\|u\|_p = \|u\|_{L^p(\Omega;E)} = \left(\int_{\Omega} \|u(x)\|_E^p \, dx \right)^{\frac{1}{p}}, \ 1 \leq p < \infty,$$

$$\|u\|_{L^\infty(\Omega;E)} = ess \sup_{x \in \Omega} \|u(x)\|_E .$$

Here, $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{C}$ denote the set of real, entire and complex numbers, respectively. Let $f = f(u) = f(u)(x)$ be a composite function for E-valued function

u. Assume the function $u : \mathbb{R}^n \rightarrow E$ is such that $f(u)(x) \in E$ for $x \in \mathbb{R}^n$. We consider the action of $f(u)(x)$ on the abstract Sobolev spaces. Let $0 \leq s < \infty$ and

$$m = \left\{ \begin{array}{c} s\text{, if is an integer } [s]\,, \\ [s] + 1\text{, otherwise.} \end{array} \right.$$

The E-valued function $f(u)$ said to be s-admissible if $f(0) = 0$ and is Fr´echet differentiable in E with

$$M = \max_{|\beta|=k} \sup_{x \in \mathbb{R}^n} \left\| D_x^\beta f(u)(x) \right\|_E < \infty, \ \ \beta = (\beta_1, \beta_2, ..., \beta_n)\,,$$

$$D_x^\beta = \frac{\partial^\beta}{\partial x_1^{\beta_1} \partial x_2^{\beta_2} ... \partial x_n^{\beta_n}}\,,$$

where the maximum is taken over $k \in \{1, 2, ...m\}$.

$S(\mathbb{R}^n; E)$ denotes an E-valued Schwartz class, i.e. the space of all E-valued rapidly decreasing smooth functions on $\mathbb{R}^n$ equipped with its usual topology generated by seminorms. $S(\mathbb{R}^n; \mathbb{C})$ is denoted by $S(\mathbb{R}^n)$. Here, $S'(\mathbb{R}^n) = S'(\mathbb{R}^n; E)$ denote the space of all continuous linear operators from $S(\mathbb{R}^n)$ into E equipped with the bounded convergence topology. Recall $S(\mathbb{R}^n; E)$ is norm dense in $L^p(\mathbb{R}^n; E)$, when $1 \leq p < \infty$.

Here, $\mathbb{F}$ denotes the Fourier transform. Let $L^{s,p}(\mathbb{R}^n; E)$ denotes E-valued Bessel space of order $s \in \mathbb{R}$, that is defined as:

$$L^{s,p}(E) = L^{s,p}(\mathbb{R}^n; E) = \{u \in S'(\mathbb{R}^n; E),$$

$$\left. \|u\|_{L^{s,p}(E)} = \left\| \mathbb{F}^{-1} \left(I + |\xi|^2 \right)^{\frac{s}{2}} \hat{u} \right\|_{L^p(\mathbb{R}^n; E)} < \infty \right\}.$$

It clear that $L^{0,p}(\mathbb{R}^n; E) = L^p(\mathbb{R}^n; E)$.

Let $\dot{L}^{s,p}(\mathbb{R}^n; E)$, $s > 0$, $p \in (1, \infty)$ be the E-valued Riesz potential space, i.e. the homogeneous counterpart to the inhomogeneous space $L^{s,p}(\mathbb{R}^n; E)$. Note that for $s = m \in \mathbb{Z}^+$ the space $\dot{L}^{s,p}(\mathbb{R}^n; E)$ is defined by

$$\|u\|_{\dot{L}^{s,p}(\mathbb{R}^n; E)} \approx \sum_{|\alpha| = s} \|D^\alpha u\|_{L^p(\mathbb{R}^n; E)}\,,$$

wile

$$\|u\|_{L^{s,p}(\mathbb{R}^n; E)} \approx \sum_{|\alpha| \leq s} \|D^\alpha u\|_{L^p(\mathbb{R}^n; E)} \approx \|u\|_{L^p(\mathbb{R}^n; E)} + \|u\|_{\dot{L}^{s,p}(\mathbb{R}^n; E)}\,.$$

Let E_0 and E be two Banach spaces and E_0 is continuously and densely embedded into E. Let $Y^{s,p}(E_0, E) = L^{s,p}(\mathbb{R}^n; E_0, E)$ denote the Bessel-Lions type spaces i.e.,

$$L^{s,p}(\mathbb{R}^n; E_0, E) = \{u \in L^{s,p}(\mathbb{R}^n; E) \cap L^p(\mathbb{R}^n; E_0)\,,$$

$$\|u\|_{L^{s,p}(\mathbb{R}^n;E_0,E)} = \|u\|_{L^p(\mathbb{R}^n;E_0)} + \|u\|_{L^{s,p}(\mathbb{R}^n;E)} < \infty \Big\} .$$

$W^{m,p}(\Omega;E)$ denotes an E-valued Sobolev space with norm,

$$\|u\|_{W^{m,p}(\Omega;E)} = \|u\|_{L^p(\Omega;E)} + \sum_{k=1}^{n} \left\| \frac{\partial^m u}{\partial x_k^m} \right\|_{L^p(\Omega;E)} < \infty.$$

In a similar way, we define the following Sobolev-Lions type spaces,

$$W^{m,p}(\Omega;E_0,E) = W^{m,p}(\Omega;E) \cap L^p(\Omega;E_0).$$

Let $s = m + \sigma$, m integer, $0 < \sigma < 1$. Consider E-valued Sobolev-Slobodetskii space $W^{s,p}(\mathbb{R}^n;E)$ defined by

$$\|u\|^p_{W^{s,p}(\mathbb{R}^n;E)} \sim \|u\|^p_{X_p} + \|D^m u\|^p_{X_p} +$$

$$\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \frac{\|D^m u(x) - D^m u(y)\|^p}{|x-y|^{n+\sigma p}} dxdy < \infty.$$

We start by recalling the Littlewood-Paley decomposition of temperate distributions in vector valued function spaces. In order to define abstract Lizorkin-Triebel spaces we consider the dyadic-like subsets $\{I_k\}_{k=0}^{\infty}$ of $\mathbb{R}^n$ and partition of unity $\{\varphi_k\}_{k=0}^{\infty}$ defined e.g. in [12, § 1]. For $u \in S'$, we set $u_k = u * \varphi_k$, where $u * \varphi_k$ denotes the convolution of the functions u and φ_k.We have $u = \sum\limits_k u_k$ in $S'(E)$. Here, $l_q(E)$-denotes the E-valued sequance space $u = \{u_k\}_{k=0}^{\infty}$ with norm given by

$$\|u\|_{l_q(E)} = \left[\sum_{k=0}^{\infty} \|u_k(x)\|^q_E \right]^{\frac{1}{q}}, \ 1 \le q < \infty, \ \|u\|_{l_\infty(E)} = \sup_k \|u_k(x)\|_E .$$

Let $-\infty < s < \infty$ and $0 < p,\ q \le \infty$. The E-valued Lizorkin-Triebel space $F^s_{p,q}(E) = F^s_{p,q}(\mathbb{R}^n;E)$ is the set of all $f \in S'(\mathbb{R}^n;E)$ for which

$$\|f\|_{F^s_{p,q}(\mathbb{R}^n;E)} = \left\| \left\{ 2^{ks} \left(\check{\varphi}_k * u \right) \right\}_{k=0}^{\infty} \right\|_{L_p(\mathbb{R}^n;l_q(E))} =$$

$$\begin{cases} \left\| \left\| \left\{ 2^{ks} u_k(x) \right\} \right\|_{l_q(E)} \right\|_{L^p(\mathbb{R}^n)} < \infty, \text{ if } 1 \le p < \infty, \\ \sup\limits_{x \in \mathbb{R}^n} \left[\left\| \left\{ 2^{ks} u_k(x) \right\} \right\|_{l_q(E)} \right] < \infty, \text{ if } p = \infty. \end{cases}$$

$F^s_{p,q}(\mathbb{R}^n;E)$-together with the norm in (1.3) is a Banach space (see e.g. [12, § 10]. It can be shown (see [12, § 11) that different choices of $\{\varphi_k\}$ lead to equivalent norms on $F^s_{p,q}(\mathbb{R}^n;E)$ for $E = \mathbb{C}$.

Moreover, by using the Fourier multipler theorems in $L^p(\mathbb{R}^n;E)$, we get that the spaces $W^{s,p}(\mathbb{R}^n;E)$ also coincide with the E-valued Besov spaces

$B_{p,p}^{s}\left(\mathbb{R}^{n};E\right)$. But for $p\neq 2$, the spaces $W^{s,p}\left(\mathbb{R}^{n};E\right)$ do not coincide with the Bessel spaces $L^{s,p}\left(\mathbb{R}^{n};E\right)$.

Let

$$X_{p}=L^{p}\left(\mathbb{R}^{n};E\right),\ X^{s,p}=X^{s,p}\left(E\right)=L^{s,p}\left(\mathbb{R}^{n};E\right),\ \overset{\circ}{X}{}^{s,p}=\dot{L}^{s,p}\left(\mathbb{R}^{n};E\right).$$

$$X^{s,p}\left(E_{0},E\right)=W^{s,p}\left(\mathbb{R}^{n};E_{0},E\right),\ Y_{p}=Y_{p}\left(E\right)=L^{p}\left(\mathbb{R}_{T}^{n};E\right),$$

$$Y^{s,p}=W^{s,p}\left(\mathbb{R}^{n};E\right),\ Y^{m,s,p}=Y^{m,s,p}\left(E_{0},E\right)=W^{m,s,p}\left(\mathbb{R}_{T}^{n};E_{0},E\right).$$

Let X and Y be two Banach spaces. $\left(X,Y\right)_{\theta,p}$ for $\theta\in(0,1)$, $p\in[1,\infty]$ denotes the real interpolation spaces defined by K-method [12, §1.3.2]. $L\left(X,Y\right)$ will denote the space of all bounded linear operators from X to Y. For $Y=X$ it will be denoted by $L\left(X\right)$.

Here,

$$S_{\phi}=\left\{\lambda\in\mathbb{C},\ \left|\arg\lambda\right|\leq\phi,\ 0\leq\phi<\pi\right\}.$$

A closed linear operator A is said to be ϕ-dissipative (or dissipative) in a Banach space X with bound $M>0$ if $D\left(A\right)$ and $R\left(A\right)$ are dense on E, $N\left(A\right)=\{0\}$ and

$$\left\|\left(A-\lambda I\right)^{-1}\right\|_{L(X)}\leq M\left|\lambda\right|^{-1}$$

for any $\lambda\in S_{\phi}$, $0\leq\phi<\pi$, where I is the identity operator in X, $D\left(A\right)$ and $R\left(A\right)$ denote domain and range of the operator A.

Definition 1.3. A Banach space E has Fourier type $r\in[1,2]$ provided the Fourier transform $\mathbb{F}$ defines a bounded linear operator from $L^{r}(\mathbb{R}^{n};E)$ to $L^{r^{'}}\left(\mathbb{R}^{n};E\right)$ for $\frac{1}{r}+\frac{1}{r^{\prime}}=1$ (see e.g [5, Remark 2.3]).

From [5] we obtain:

Proposition 1.1. A Banach space E has Fourier type $r\in[1,2]$. Assume an operator-function Ψ belongs to $B_{r,1}^{\frac{n}{r}}\left(\mathbb{R}^{n};L\left(E\right)\right)$. Then Ψ is a Fourier multiplier in $L^{p}\left(\mathbb{R}^{n};E\right)$ for $p\in\left[1,\infty\right]$.

Consider the following Cauchy problem

$$u^{'}\left(t\right)=Au\left(t\right)+f\left(t\right),\ u\left(0\right)=0,\ t\in 0\in\left(0,T\right),\tag{1.1}$$

where A is a linear operator in a Banach space E.

Let $X=L^{p}(\mathbb{R};E)$. By reasoninig as in [11, Theorem 4.3] we prove the following result.

Theorem $\mathbf{A}_{0}$. Assume that a Banach space E has Fourier type $r\in\left(\ 1,2\ \right]$ and $1\leq p\leq\infty$. Let A be a ψ-sectorial operator in E with $\psi>\frac{\pi}{2}$. Then for all $f\in X$ there exists a unique solution of the problem (1.1) and the following maximal regularity estimate holds

$$\left\|u^{'}\right\|_{X}+\left\|Au\right\|_{X}\leq C\left\|f\right\|_{X}.\tag{1.2}$$

Assumption $\mathbf{A}_{0}$. Assume that $f^{(j)}\left(u\right)$ is a continuous function in $u\in W^{s,p}\left(\mathbb{R}^{n};E\right)$. Let Λ be a subset of $W^{s,p}\left(\mathbb{R}^{n};E\right)\cap L^{\infty}\left(\mathbb{R}^{n};E\right)$ such that $f^{(j)}\left(u\right)\left(x\right)\in E$ for $u\in\Lambda$ and $x\in\mathbb{R}^{n}$ for $j=1,2,...,m$. Here, m is a positive integer.

First, we show the following:

Theorem 1.1. Let the Assumption A_0 hold. Assume that $s > 1$, $p \in (1, \infty)$, and f is s-admissible. If $u \in \Lambda \cap \dot{X}^{1,sp}$, then $f(u) \in X^{s,p} \cap \dot{X}^{1,sp}$ and

$$\|f(u)\|_{X^{s,p}} \leq M \left(\|u\|_{X^{s,p}} + \|u\|_{\dot{X}^{1,sp}}^{s}\right), \tag{1.3}$$

$$\|f(u)\|_{\dot{X}^{1,sp}} \leq M \|u\|_{\dot{X}^{1,sp}}. \tag{1.4}$$

Remark 1.1. For $0 < s \leq 1$ and $1 < p < \infty$, $f(u) \in X^{s,p}$ for all $u \in X^{s,p}$ and any s-admissible f. In fact, we see below (in Section 2) that in this case, we have

$$\|f(u)\|_{X^{s,p}} \in M \|u\|_{X^{s,p}}.$$

So, in particular, it is easy to see that (1.3) holds.

We also have $f(u) \in X^{s,p}$ for all $u \in X^{s,p}$, when $s \geq \frac{n}{p}$ as a consequence of Theorem 1 since the imbedding theorem in E-valed Sobolev spaces (see e.g. [6, 10]) implies that $X^{s,p}$ is continuously imbedded into $\dot{X}^{1,sp}$ whenever $sp \geq n$, $s > 1$ and $1 < p < \infty$.

However, for a $s \in Z^{+}$, Dahlberg [9] even in a scalar case (i.e. $E = \mathbb{C}$) has shown that if $1 < s < n/p$, $1 < p < \infty$, $f(u) \in X^{s,p}$ for all $u \in X^{s,p}$ and any for s-admissible f, then $f(u) = cu$ for some $c \in \mathbb{R}$.

In view of Dahlberg's negative result, a natural question is to determine what additional conditions on $u \in X^{s,p}$ and on the space E guarantee that $f(u) \in X^{s,p}$ for s-admissible f. The first result of this type (for case of $E = \mathbb{C}$) is obtained from the Gagliardo-Nirenberg lemma which implies that $f(u) \in X^{s,p}$ for every $u \in X^{s,p} \cap X^{\infty}$.

In this regard, we prove the following result:

Theorem 1.2. Let $s = m \in Z^{+}$ and $m \geq 1$. Assume that the Assumption A_0 is satisfied. Suppose $f(u) \in X^{m,p}$ for all $u \in \Phi \subset X^{m,p}$ and any m-admissible f. Then $u \in X^{m,p} \cap \dot{X}^{1,mp}$.

Remark 1.2. For $0 \leq s \leq 1$ and $p \in (1, \infty)$, $f(u) \in X^{s,p}$ for all $u \in X^{s,p}$ and any s-admissble f.

Here, $X_{,j,p}$ denotes a real interpolation spaces between $X^{s,p}$ and X_p, i.e.

$$X_{,j,p} = \left(X^{s,p}, X_p\right)_{\theta_j,p}, \ \theta_j = \frac{jp+1}{mp}, \ j = 0, 1, ..., m-1.$$

Remark 1.3. By definition of the space $Y^{m,s,p}(E_0, E)$, we have

$$Y^{m,s,p}(E_0, E) = W^{m,p}\left(0, T; X^{s,p}(E_0, E), X_p(E)\right).$$

Then, in virtue of J. Lions-J. Peetre trace result (see e.g [12, § 1.8]) the map $u \to u^{(j)}(t_0)$, $t_0 \in [0, T]$ is continuous and surjective from $Y^{m,s,p}(E_0, E)$ onto $X_{,j,p}$ and there is a constant C_1 such that

$$\left\|u^{(j)}(t_0)\right\|_{X_{,j,p}} \leq C_1 \|u\|_{Y^{m,s,p}(E_0,E)}, \ 1 \leq p \leq \infty. \tag{1.5}$$

Assume $s = m + \sigma$, m integer and $0 < \sigma < 1$. Let $Y^{s,p} = Y^{s,p}(E)$ denotes the Sobolev-Slobodetskii space $W^{s,p}(\mathbb{R}^n; E)$.

We prove here, the following composition properties in E-valued Sobolev-Slobodetskii space $Y^{s,p}$:

Theorem 1.3. Let the Assumption A_0 hold. Assume that $s>1$, $p\in(1,\infty)$, and f is s-admissible. Then the map $u\rightarrow f(u)$ is well-defined and continuous from $Y^{s,p}\cap Y^{1,sp}$ into $Y^{s,p}$.

Finally, we prove the following coposition and product properties in Sobolev -Lions spaces $Y^{m,s,p}(E_0,E)$:

Theorem 1.4. Assume that the following conditions are satisfied:

(1) E, E_0 be two Banach spaces, E_0 continuously and densely belongs into E;

(2) $f=f(t,x,u)$ is s-admissible with $s\geq 0$, $x\in\mathbb{R}^n$ for a.a. $t\in[0,T]$;

(3) the function $u\rightarrow f(t,x,u)$: $\mathbb{R}_T^n\times X_0\rightarrow E$ is a measurable in $(t,x)\in\mathbb{R}_T^n$ for $u\in X_0$;

(4) $f(t,x,u)$ is continuous in $u\in X_0$ and $f(t,x,.)\in C^{[s]+1}(X_0;E)$ uniformly with respect to $(t,x)\in\mathbb{R}_T^n$.

Then for any $u\in Y^{m,s,p}(E_0,E)$, we have $f(u(t_0,.))(.)\in X^{s,p}$. Moreover, for all $u\in Y^{m,s,p}(E_0,E)$ the following estimate holds

$$\|f(u(t_0,.))\|_{X^{s,p}}\lesssim\|u)\|_{X^{s,p}}\,. \tag{1.6}$$

Remark 1.6. The conditions of Theorem 1.4 does not includes the Assumption A_0. This is due to fact that here, we used the trace result (1.5) instead of it.

As an application of Theorems 1.1-1.4 consider now the Cauchy problem for nonlocal abstract diffusion equation,

$$\partial_t u-a\Delta u-A*u=f(u)\,,\ (t,x)\in(0,T)\times\mathbb{R}^n, \tag{1.7}$$

$$u(0,x)=\varphi(x)\ \text{ for a.e. } x\in\mathbb{R}^n,$$

where $A=A(x)$ and $f(u)$ is a nonlinear operator functions in a Banach space E, respectively, $u=u(t,x)$ is a E-valued unknown function, a is a complex number, $T\in(0,\ \infty]$, $f(u)$ is a given nonlinear function and $\varphi(x)$ is a given E-valued functions.

Here, we derive the existence, uniqueness, and L^p-regularity properties to solution of the problem (1.7).

We use here, the equivalent Littlewood-Paley characterization of $\dot{L}^{s,p}(\mathbb{R}^n;E)$ for its definition (see [1] for $E=\mathbb{C}$). Let ϕ be a finite function defined in [1]. For $\nu\in\mathbb{Z}$, set $\phi_\nu(x)=2^{\nu n}\phi(2^\nu x)$. For $p\in(1,\infty)$ and $s\geq 0$, let

$$\|u\|_{\dot{L}^{s,p}(\mathbb{R}^n;E)}=\left\|\sum_{\nu\in\mathbb{Z}}\|2^{\nu s}\phi_\nu*u\|_E\right\|_{L^p}.$$

Note that $\dot{L}^{s,p}(\mathbb{R}^n;E)=L^{s,p}(\mathbb{R}^n;E)$ for $s=0$, $p\in(1,\infty)$ by Littlewood-Paley theory (see e.g. [28] for $E=\mathbb{C}$).

For u, $\upsilon>0$ the relations $u\lesssim\upsilon$, $u\approx\upsilon$ means that there exist positive constants C, C_1, C_2 independent on u and υ such that, respectively

$$u\leq C\upsilon,\ C_1\upsilon\leq u\leq C_2\upsilon.$$

2. Preliminaries

Let E be a Banach space and $f(x)$ is a E-valued function. Sometimes we will denote $\|f(x)\|_E$ just by $\|f(x)\|$. For proving the main results in a similar way as in [1, Theorem A], we get the following lemmas:

Lemma 2.1. Assume that $0 \leq \alpha_1 < \alpha_2 < \infty$, p_1, $p_2 \in (1, \infty)$, $\theta \in (0,1)$, $\alpha = (1-\theta)\alpha_1 + \theta\alpha_2$, and $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$. Then

$$\|u\|_{\mathring{X}^{\alpha,p}} \leq \|u\|_{\mathring{X}^{\alpha_1,p_1}}^{1-\theta} \|u\|_{\mathring{X}^{\alpha,p}}^{\theta}. \tag{2.1}$$

Lemma 2.2. Suppose that $\theta \in (0,1)$, $p \in (1,\infty)$ and $\alpha > 0$. Then

$$\|u\|_{\mathring{X}^{\alpha\theta,p\backslash\theta}} \leq \|u\|_{BMO(E)}^{1-\theta} \|u\|_{\mathring{X}^{\alpha,p}}^{\theta}. \tag{2.2}$$

Lemma 2.3. Suppose that $\theta \in (0,1)$, $p \in (1,\infty)$ and $\alpha > 0$. If $u \in X^q$ for some $q \in (1,\infty)$, then

$$\|u\|_{\mathring{X}^{\alpha,p}} \approx \|S_\alpha u\|_{X_p} \tag{2.3}$$

and if $1 \leq t < p$,

$$\|D_t^\alpha u\|_{X_p} \leq C(\alpha, p, t) \|u\|_{\mathring{X}^{\alpha,p}}. \tag{2.4}$$

Lemma 2.4. Suppose that $\theta \in (0,1)$, $p \in (1,\infty)$, $k \in Z$, $k \geq 2$. Let $p_i \geq 1$, $\gamma_i \geq 1$, $i = 1,2,...,k$, and $\frac{1}{p_j} + \sum\limits_{i \neq j} \frac{1}{\gamma_i} = 1$, for $j = 1,2,...,k$. Then

$$\left\|\prod_{i=1}^{k} u_i\right\|_{\mathring{X}^{\theta,p}} \leq C \sum_{j=1}^{k} \|u_j\|_{\mathring{X}^{pp_j}} \prod_{i=1, i\neq j}^{k} \|u_i\|_{X^{\gamma_i p}}.$$

Let $f(u) = f(u)(x)$. For proving of the main theorems we need the following

Theorem 2.1. Let the Assumption A_0 hold, $s \geq 0$, $f \in C^{[s]+1}(\mathbb{R}^n; E)$ with $f(0) = 0$. Then for any $u \in \Phi$, we have

$$f(u)(.) \in X^{s,p} \cap X_\infty.$$

Moreover, there is some constant $A(M)$ depending on M such that for all $u \in \Phi$ with $\|u\|_{X_\infty} \leq M$,

$$\|f(u)\|_{X^{s,p}} \leq C(M) \|u)\|_{X^{s,p}}. \tag{2.5}$$

Proof. By Assumption A_0, $f(u) \in E$ for $u \in \Phi$. For $s = 0$ in view of $f(0) = 0$, we get

$$f(u) = \int_0^1 f^{(1)}(\sigma u)\, d(\sigma u).$$

It follows that

$$\|f(u)\|_{X_p} \leq C(M) \|u\|_{X_p}.$$

If s is a positive integer, we have

$$\|f(u)\|_{X^{s,p}} \leq C\left[\|f(u)\|_{X_p} + \sum_{k=1}^{n}\left\|\frac{\partial^s}{\partial x_k} f(u)\right\|_{X_p}\right]. \tag{2.6}$$

By calculation of derivative and applying Holder inequality, Gagliardo-Nirenberg's inequality in E-valued X_p spaces, we have

$$\left\|\frac{\partial^{\beta_k} u}{\partial x_i}\right\|_{X_{p_k}} \leq C\,\|u\|_{X_\infty}^{1-\frac{\beta_k}{l}}\left\|\frac{\partial^s u}{\partial x_i^s}\right\|_{X_p}^{\frac{\beta_k}{l}}. \tag{2.7}$$

Hence, from (2.6) and (2.7) we get

$$\left\|\frac{\partial^s}{\partial x_i} f(u)\right\|_{X_p} \leq C\,(M)\left\|\frac{\partial^s u}{\partial x_i^s}\right\|_{X_p}. \tag{2.8}$$

Then combining (2.7) and (2.8) we obtain (2.5).

Let s is not integer number and $m = [s]$. From the above proof, we have

$$\|f(u)\|_{X^{m,p}} \leq C\,(M)\,\|u)\|_{X^{m,p}}\,,\ \|f(u)\|_{X^{m+1,p}} \leq C\,(M)\,\|u)\|_{X^{m+1,p}}\,.$$

Then from the first part of the proof, we get the estimate (2.5)

3. Proofs of main theorems

Proof of Theorem 1.1. Since $f\,(0) = 0$, we have

$$\|f\,(u)\,(x)\| = \|f\,(u)\,(x) - f\,(0)\| \leq \|f\|_{X_\infty}\,\|u\,(x)\|\,.$$

Hence,

$$\|f\,(u)\|_{X_p} \leq \|f\|_{X_\infty}\,\|u\|_{X_p}\,. \tag{3.1}$$

Then $\frac{\partial}{\partial x_i} f\,(u)\,(x) = f^{(1)}\,(u)\,\frac{\partial u}{\partial x_i}$ for $i = 1, 2, ..., n$. Hence, by (2.5)

$$\|f\,(u)\|_{\dot{X}^{1,p}} \leq C\left\|f^{(1)}\right\|_{X_\infty}\|u\|_{\dot{X}^{1,p}}\,,\ p \in (1,\infty)\,. \tag{3.2}$$

If $\alpha \in (0,1)$, applying the mean value theorem to $f\,(u\,(x+ry)) - f\,(u\,(x))$ we have

$$S_\alpha f\,(u) \leq \left\|f^{(1)}\right\|_{X_\infty} S_\alpha u.$$

Hence, by (2.3),

$$\|f\,(u)\|_{\dot{X}^{\alpha,p}} \leq C\left\|f^{(1)}\right\|_{X_\infty}\|u\|_{\dot{X}^{\alpha,p}}\,,\ p \in (1,\infty)\,. \tag{3.3}$$

Now, by reasoning as in [1, Theorem A], we obtain the assertion.

Proof of Theorem 1.2. First notice that $u \in X^{m,p}$, since $f\,(t) = t$ is m–admissible for any m. Let $\chi \in C^\infty\,(\mathbb{R}^n)$ satisfy supp$\chi \in [-1,1]$ and $\chi\,(t) = 1$ for $t \in \left[-\frac{2}{3}, \frac{2}{3}\right]$. Then by reasononing as in [1, Theorem B], we get the assertion.

Proof of Theorem1.3. For proving given theorem, we need the following lemmas:

By reasoning as in [16], we have

Lemma 3.0 (Abstract Gagliardo-Nirenberg's inequality). Let E be a Fourier type space. Assume that $u \in L^p(\mathbb{R}^n; E)$, $D^m u \in L^q(\mathbb{R}^n; E)$, $p, q \in (1, \infty)$. Then for i with $0 \leq i \leq m$, $m > \frac{n}{q}$ we have

$$\left\| D^i u \right\|_r \leq C \left\| u \right\|_p^{1-\mu} \sum_{k=1}^{n} \left\| D_k^m u \right\|_q^{\mu}, \tag{3.9}$$

where

$$\frac{1}{r} = \frac{i}{m} + \mu \left(\frac{1}{q} - \frac{m}{n} \right) + (1 - \mu) \frac{1}{p},\ \frac{i}{m} \leq \mu \leq 1.$$

Note that, for $E = \mathbb{C}$ the lemma considered by L. Nirenberg [7].

By reasoning as in [13], we get the following

Lemma 3.2. Let the Assumption A_0 hold and $-\infty < s < \infty$, $0 < p$, $q < \infty$. For every $j \geq 0$, let $f_j \in S'(\mathbb{R}^n; E)$ be such that supp $f_j \subset B_{2^{j+2}}$. Then

$$\left\| f_j \right\|_{F_{p,q}^s(E)} \lesssim \left\| \left\| \left\{ 2^{sj} f_j \right\}_{k=0}^{\infty} \right\|_{l_q(E)} \right\|_{L^p(\mathbb{R}^n)}. \tag{3.10}$$

Lemma 3.3. Let E be a Fourier type space. For any $f \in L^p(\mathbb{R}^n; E)$, $p \in (1, \infty)$,

(1)

$$\left\| Mf \right\|_{L^p(\mathbb{R}^n; E)} \lesssim \left\| f \right\|_{L^p(\mathbb{R}^n; E)};$$

(2) for any sequence of function$\{f_j\}$,

$$\left\| \left\| \left\{ Mf_j(x) \right\} \right\|_{l_q(E)} \right\|_{L^p(\mathbb{R}^n)} \lesssim \left\| \left\| \left\{ f_j(x) \right\} \right\|_{l_q(E)} \right\|_{L^p(\mathbb{R}^n)};$$

(3) for any fixed $\varphi \in S(\mathbb{R}^n)$ and any function f,

$$\left\| f * \varphi \right\|_E \lesssim Mf(x), \text{ for } t > 0,\ x \in \mathbb{R}^n.$$

Lemma 3.4. For $0 < s_1 < s_2 < \infty$, $1 < p < \infty$, $0 < q \leq \infty$, so that $s = \theta s_1 + (1 - \theta) s_2$, $\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$, we have

$$\left\| f \right\|_{F_{\frac{p}{\theta}, q}^{\theta s}} \lesssim \left\| f \right\|_{X^{s,p}}^{\theta} \left\| f \right\|_{X_\infty}^{1-\theta}. \tag{3.11}$$

By reasoning of as in the Runst-Sickel lemma [9, p.345], we obtain the same estimates for E-valued Lizorkin-Triebel spaces given by:

Lemma 3.5. Let the Assumption A_0 hold and $-\infty < s_1 < s_2 < \infty$, $0 < r_1$, $r_2 \leq \infty$, $0 < p_1$, $p_2 \leq \infty$, $0 < q \leq \infty$ such that

$$0 < \frac{1}{p} = \frac{1}{p_1} + \frac{1}{r_1} = \frac{1}{p_2} + \frac{1}{r_2} < 1.$$

Then for $f \in F^s_{p_1,q}(E) \cap X_{r_1}$ and $g \in F^s_{p_2,q}(E) \cap X_{r_2}$ the following estimate holds

$$\|fg\|_{F^s_{p,q}(E)} \lesssim \left\| Mf \left\| \{2^{sj} g_j(x)\} \right\|_{l_q(E)} \right\|_{L^p(\mathbb{R}^n)} +$$

$$\left\| Mg \left\| \{2^{sj} f_j(x)\} \right\|_{l_q(E)} \right\|_{L^p(\mathbb{R}^n)} \tag{3.12}$$

and

$$\|fg\|_{F^s_{p,q}(E)} \lesssim \|f\|_{F^s_{p_1,q}(E)} \|g\|_{X_{r_2}} + \|g\|_{F^s_{p_2,q}(E)} \|f\|_{X_{r_1}}. \tag{3.13}$$

Lemma 3.6. Let $-\infty < s_1 < s_2 < \infty$, $0 < q_1$, $q_2 \leq \infty$, $0 < p_1$, $p_2 \leq \infty$, $0 < q \leq \infty$, $0 < \theta < 1$, and define

$$s = s_1\theta + s_2(1-\theta), \; \frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}.$$

Then the following estimate holds

$$\|f\|_{F^s_{p,q}(E)} \lesssim \|f\|^{\theta}_{F^{s_1}_{p_1,q_1}(E)} \|f\|^{1-\theta}_{F^{s_2}_{p_2,q_2}(E)}. \tag{3.14}$$

Proof of Theorem 1.3. The conclusion is obtained, when s is an integer by using E-valued Gagliardo-Nirenberg inequalities. Assume s non integer. Clearly, the map $u \to f(u)$ is well defined and continuous from $Y^{s,p} \cap Y^{1,sp}$ into X_p, since $f(0) = 0$, f is Lipschitz function and the embedding $Y^{s,p} \subset X_p$ is continuous. Thus it suffices to prove that the map

$$u \to Df(u) = f^{(1)}(u)\, Du$$

is well defined and continuous from $Y^{s,p} \cap Y^{1,sp}$ into $Y^{s-1,p}$. This fact is derived as in [1, Theorem A].

Now, we will consider the Sobolev-Lions type space

$$Y^{m,s,p}(E_0, E) = W^{m,s,p}(\mathbb{R}^n_T; E_0, E).$$

Proof of Theorem 1.4. By (1.3) the maps,

$$u \to u^{(j)}(t_0, x), \; j = 0, 1, 2, ..., m-1$$

are bounded from $u \in Y^{m,s,p}(E_0, E)$ onto

$$X_{j,p} = (X^{s,p}(E_0, E), X_p)_{\theta_j,p}, \; \theta_j = \theta_j(s,p) = \frac{jp+1}{ps}.$$

Since

$$X^{s,p}(E_0, E) = X^{s,p}(E) \cap X_p(E_0),$$

by properties of real interpolation of Banach spaces, interpolation of the intersection of the spaces (see e.g. [12, §1.3]), and in view of definition $X^{s,p}(E_0, E)$, we obtain

$$X_{j,p} = (X^{s,p}(E_0, E) \cap X_p(E_0), X_p)_{\theta_j,p} = L^{s(1-\theta_j),p}\left(\mathbb{R}^n; (E_0, E)_{\theta_j,p}, E\right).$$

Since the embedding

$$X^{s,p}(E_0,E)\subset X^{s,p}(E)$$

is continuous, by virtue of Theorem 1.1 and trace result (1.3) for any $u\in Y^{s,p}(E_0,E)$, we get that $f(u(t_0,.))\in X^{s,p}(E)$ and the estimate (1.4) holds.

4. Regularity properties of abstract diffision equations

In this section, by using Theorems 1.4, we derive the existence, uniqueness, and regularity properties of the problem (1.7). The abstract evolution equations studied e.g. in [10, 11] and the references therein. In contrast to the mentioned works, we will study the existence, uniqueness, and L^p-regularity properties of the parabolic problem (1.7). Consider, first the corresponding linear problem

$$u_t-a\Delta u-A*u=g(x,t)\,,\ x,\ t\in\mathbb{R}_T^n,\ T\in(0,\infty]\,, \tag{4.1}$$

$$u(0,x)=\varphi(x)\ \text{ for a.e. } x\in\mathbb{R}^n,\ \mathbb{R}_T^n=(0,T)\times\mathbb{R}^n,$$

where $A=A(x)$ is a linear operator function defined in a Hilbert space E and a is a complex number, generally.

Let $\hat{A}(\xi)$ be the Fourier transformation of $A(x)$, i.e. $\hat{A}(\xi)=F(A(x))$. We assume that $\hat{A}(\xi)$ is uniformly dissipative operator in E. Let

$$\eta=\eta(\xi)=-\hat{A}+a|\xi|^2 I,\ U(t)=U(\xi,t)=e^{t\eta(\xi)},$$

where I is an identity operator in space E.

Condition 4.1. Assume: (1) E is a Fourier space of type $r\in[1,2]$; (2) $A=A(x)$ is a linear operator with domain $D(A)$ independent on $x\in\mathbb{R}^n$ such that $Au\in L^1(\mathbb{R}^n;E)$ for $u\in S(\mathbb{R}^n;D(A))$ and $\hat{A}(\xi)$ is a uniformly dissipative operator that generates a strongly continuous and uniformly bounded semigroup $U(\xi,t)$ in E; (3) $a\in\mathbb{C}$ such that $a+\lambda\in\bar{S}(\phi)$ for all $\lambda\in\bar{S}(\phi)$ and $\phi>\frac{\pi}{2}$; (4) $\hat{A}(\xi)$ is a differentiable operator function with independent of ξ domain with $D\left(D_\xi^\alpha\hat{A}(\xi)\right)=D\left(\hat{A}\right)=D(A)$ for $\alpha=(\alpha_1,\alpha_2,...,\alpha_n)$ and $|\alpha|\leq n$.

Let

$$X_{0p}=\left(X^{s,p},X_p\right)_{\frac{1}{p},p}\,,\ X_{0p}(A)=\left(X^{s,p}(A)\,,X_p(A)\right)_{\frac{1}{p},p}\,,$$

$$1<p<\infty.$$

By reasoning as in [10], by using Theorem A_0 we show that the problem (4.1) has a solution

$$u(x,t)=S(t)\varphi+D(t)g, \tag{4.2}$$

where $S(t)$ and $D=D(t)$ are linear operator functions defined by

$$S(t)\varphi=\mathbb{F}^{-1}\left[U(\xi,t)\hat{\varphi}(\xi)\right],\ Dg=F^{-1}\tilde{D}(\xi,t)g, \tag{4.3}$$

$$\tilde{D}(\xi,t)\,g=\int_0^t \left[U(\xi,t-\tau)\,\hat{g}(\xi,\tau)\right]d\tau.$$

By using the Proposition 1.1, and by reasoning as in [24] we have the following results:

Theorem 4.1. Let the Condition 4.1 hold, $0\leq\gamma<1-\frac{1}{p}$ and

$$s>\frac{pn}{p-1}\left(\frac{2}{r}+\frac{1}{p}\right) \tag{4.4}$$

for $p\in(1,\infty)$ and a $r\in(\,1,2\,]$. Then for $\varphi\in X_{0p}\left(A^{1+\gamma}\right)\cap X_1\left(A^{1+\gamma}\right)$, $g(.,t)\in Y_1^{s,p}(A^\gamma)$, $t\in[0,T]$ the problem (4.1) has a unique strong solution $u(x,t)\in C^{(1)}([0,T];X_\infty(A))$. Moreover, the following uniform estimate holds

$$\|A^\gamma * u\|_{X_\infty}+\|A^\gamma * u_t\|_{X_\infty}\leq C_0\,[\,\left\|A^{1+\gamma}*\varphi\right\|_{\mathbb{E}_{0p}}+\left\|A^{1+\gamma}*\varphi\right\|_{X_1}+ \tag{4.5}$$

$$\int_0^t\left(\|A^\gamma g(.,\tau)\|_{Y^{s,p}}+\left\|A^{1+\gamma}*g(.,\tau)\right\|_{X_1}\right)d\tau\,].$$

Theorem 4.2. Let the Condition 4.1 hold, $0\leq\gamma<1-\frac{1}{p}$ and (4.4) be satisfied. Moreover, for $\varphi\in Y^{s,p}(A)$, $g\in Y^{s,p}(A)$, $g(.,t)\in Y^{s,p}(A)$ the following uniform estimate holds

$$\|A^\gamma * u\|_{Y^{s,p}}+\|A^\gamma * u_t\|_{Y^{s,p}}\leq \tag{4.6}$$

$$C_0\,[\,\|A*\varphi\|_{Y^{s,p}}+\int_0^t\|A*g(.,\tau)\|_{Y^{s,p}}\,d\tau\,].$$

Consider now, nonlinear problem (1.7). For the study of the nonlinear problem (1.7) we need the following: Let $Y_0=Y^{s,p}(A,E)\cap L_\infty(A)$. Here, $Y(T)$ is the space defined by

$$Y(T)=\{\,u\in C^1(0,T;Y_0)\,,$$

$$\|u\|_{Y(T)}=\max_{0\leq t\leq T}\|A*u\|_{Y^{s,p}}+\|A*u\|_{X_\infty}<\infty\,\}.$$

It is easy to see that $Y(T)$ is a Banach space. For $\varphi\in Y_1^{s,p}(A)$, let $M=\|A*\varphi\|_{Y^{s,p}}+\|A*\varphi\|_{X_1}$.

Condition 4.3. Assume:

(1) the Condition 4.1 and Assumption A_0 are hold, $\varphi\in Y^{s,p}(A)\cap X_1(A)$ and $s>\frac{pn}{p-1}\left(\frac{2}{r}+\frac{1}{p}\right)$ for $p\in(1,\ \infty]$ and a $r\in(\,1,2\,]$;

(2) the function $u\rightarrow f(x,t,u)$: $\mathbb{R}_T^n\times X_{0,p}\rightarrow E$ is a measurable in $(x,t)\in\mathbb{R}_T^n$ for $u\in X_{0,p}$. Moreover, $f(x,t,u)$ is continuous in $u\in X_{0,p}$ and $f(x,t,.)\in C^{[s]+1}(X_{0,p};E)$ uniformly with respect to $x\in\mathbb{R}^n$, $t\in[0,T]$.

Main aim of this section is to prove the following result:

Theorem 4.3. Let the Condition 4.3 hold. Then problem (1.7) has a unique strong solution $u \in C^{(1)}([0,\ T_0);Y_0)$, where T_0 is a maximal time interval dependent of M. Moreover, if

$$\sup_{t\in[0,\ T_0)}\left(\|A*u\|_{Y_0}+\|A*u_t\|_{Y_0}\right)<\infty$$

then $T_0=\infty$.

Proof. First, we are going to prove the existence and the uniqueness of the local strong solution of (1.7) by contraction mapping principle. By (4.5), ((4.6)) the problem of finding a solution u of (1.7) is equivalent to finding a fixed point of the mapping of $G(u)$ defined by

$$G(u)=G(u)(x,t)=S(t)\varphi+\Phi(u),$$

where

$$\Phi(u)=\int_0^t \mathbb{F}^{-1}\left[U(\xi,t-\tau)\hat{f}(u)(\xi,\tau)\right]d\tau.$$

Let

$$Q=Q(M;T)=\left\{u:u\in L^p\left(\mathbb{R}_T^n;H(A)\right),\ \|u\|_{Y_0}\leq M+1\right\}$$

with T and M to be determined. So, we will find T and M so that $G(u)$ is a contraction on $Q(M;T)$. From Lemma 3.1 we know that $\Phi(u)\in L^p\left(0,T;Y_\infty^{s,p}\right)$ for any $T>0$. From Lemma 3.1 it is easy to see that the map $G(u)$ is well defined for $f\in C^{[s]+1}(X_{0p};E)$. By reasoning as in [11] we show that the operator G maps $Q(M;T)$ into $Q(M;T)$ and $G:Q(M;T)\rightarrow Q(M;T)$ is strictly contractive if T is appropriately small relative to M, i.e the map $G=G(u)$ has a unique fixed point in $Q(M;T)$. Moreover, in a similar way as in [11], we obtain the assertion.

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